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The solitary wave in water of variable depth

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Equations are derived for two-dimensional long waves of small, but finite, amplitude in water of variable depth, analogous to those derived by Boussinesq for water of constant depth. When the depth is slowly varying compared to the length of the wave, an asymptotic solution of these equations is obtained which describes a slowly varying solitary wave; also differential equations for the slow variations of the parameters describing the solitary wave are derived, and solved in the case when the solitary wave evolves from a region of uniform depth. For small amplitudes it is found that the wave amplitude varies inversely as the depth.

1. Introduction

The behaviour of surface gravity waves on a beach has been a subject of considerable theoretical and experimental research. In the simplest situation the flow is two-dimensional and irrotational, and the fluid is inviscid, incompressible and of constant density. Then, for a train of infinitesimally small amplitude oscillatory waves of frequency ω and wave-number κ, the change in amplitude α due to a gradual slope may be determined by the assumption that the rate of energy propagation remains constant (Rayleigh 1911). Thus

$$c_g \alpha^2 = \text{constant}, \tag{1.1}$$

where c_g is the group velocity,

$$c_g = \frac{d\omega}{d\kappa}, \quad \omega^2 = g\kappa \tanh \kappa h \tag{1.2}$$

is the dispersion relation, and h is the undisturbed depth. Since ω remains constant, the elimination of κ between (1.1) and (1.2) determines α as a function of h. For infinitely long waves, $\kappa h \to 0$, and this procedure leads to Green's law (Green 1837)

$$\alpha h^{\frac{1}{4}} = \text{constant}. \tag{1.3}$$

These results may also be derived by constructing an asymptotic expansion based on the assumption that if reflexion processes are ignored and the variation of h with the horizontal co-ordinate x is very small over a typical wavelength then the wave form is locally sinusoidal (Keller 1958). In addition conservation of mass requires the set up of a mean reverse flow and conservation of momentum requires a decrease in the mean depth as h decreases, both $O(\alpha^2)$ (Longuet-Higgins & Stewart 1964).

For infinitely long waves of finite amplitude, the governing equations are analogous to those of gas dynamics, and it is well known that no permanent progressing wave form is possible. However, it may be shown that a discontinuity in wave slope for a wave of elevation will cause the wave to break (i.e. the wave slope becomes infinite) before the shoreline is reached (Greenspan 1958). On the other hand if a bore reaches the shoreline in finite time, it does so with a finite speed and zero amplitude (Keller, Levine & Whitham 1960).

In this paper we shall consider the modulations formed on the Boussinesq solitary wave by a slow variation in the depth. This solitary wave is a permanent progressing wave form consisting of a simple elevation above the undisturbed surface whose amplitude α and length λ (usually defined as the width when the free surface is one-tenth of its maximum height) are such that α/h and h^2/λ^2 are comparable small quantities. It was first observed by Russell (1837), and established theoretically, to the lowest order in α/h, by Boussinesq (1871, 1872). Ippen & Kulin (1955) have performed experiments in which a solitary wave is incident on a beach of constant slope. They found that the amplitude increased with decreasing depth approximately according to the law h^{-k} where k depends on the beach slope and decreased as the slope was increased (e.g. $k = 0{\cdot}47$ for a beach slope of 0·023). In addition the wave crest became more pronounced, and there was increasing asymmetry due to steepening on the front face, as the wave climbed the beach; eventually wave breaking was observed, either due to 'peaking' at the wave crest and subsequent spilling, or due to an infinite slope on the front face and subsequent plunging.

To discuss the behaviour of a solitary wave on a beach, we first derive, in §2, equations analogous to those used by Boussinesq for the case of constant undisturbed depth. In §3 we derive various properties of the solitary wave. In §4 we consider the case when the still water depth h is a slowly varying function of the horizontal co-ordinate x and so varies little over a distance comparable with λ, the length of the wave. An asymptotic expansion is introduced, analogous to those used by Whitham (1965*a*, *b*) to discuss modulations on cnoidal waves on a constant depth, and in other situations also (we note that the solitary wave may be regarded as a limiting case of a cnoidal wave as the wave period becomes infinite). Then transport equations for the amplitude and for the other parameters determining the solitary wave are derived, either by imposing conditions which ensure that the asymptotic expansion is uniformly valid in x, or by using conservation laws. In §5 these transport equations are solved; the principal conclusion is that when the wave develops from a region where h is constant then the variation of the amplitude e_m is determined by conservation of the energy in the wave and this causes e_m, for small e_m/h, to vary as h^{-1}. Finally, in §6 the relationship of the asymptotic expansion to a certain exact solution of the governing equations is considered.

2. Equations of motion

It will be assumed that the flow is two-dimensional and irrotational, and that the fluid is inviscid, incompressible and of constant density. We shall be concerned with long waves so that if λ is a horizontal length scale for the waves, and

h_0 is a length scale for the undisturbed depth, then the parameter $\epsilon = h_0^2/\lambda^2$ is small compared to one. Since it can be anticipated that for long waves the Froude number will be close to critical we choose $(gh_0)^{\frac{1}{2}}$ as a typical velocity scale. Then introducing dimensionless co-ordinates based on λ, h_0, $(gh_0)^{\frac{1}{2}}$ we find that the equations of motion for the velocity potential $\phi(x,y,t)$ are

$$\epsilon\phi_{xx}+\phi_{yy} = 0 \quad \text{for} \quad -h < y < \eta, \tag{2.1}$$

$$\epsilon h_x\phi_x+\phi_y = 0 \quad \text{for} \quad y = -h, \tag{2.2}$$

$$\epsilon(\eta_t+\eta_x\phi_x)-\phi_y = 0 \quad \text{for} \quad y = \eta, \tag{2.3}$$

$$\epsilon(\eta+\phi_t+\tfrac{1}{2}\phi_x^2)+\tfrac{1}{2}\phi_y^2 = 0 \quad \text{for} \quad y = \eta, \tag{2.4}$$

where $y = \eta(x,t)$ is the free surface, and $y = -h(x)$ is the undisturbed depth, (q.v. figure 1). Equations (2.2), (2.3) are kinematic boundary conditions and (2.4) is the condition that the pressure be constant on the free surface.

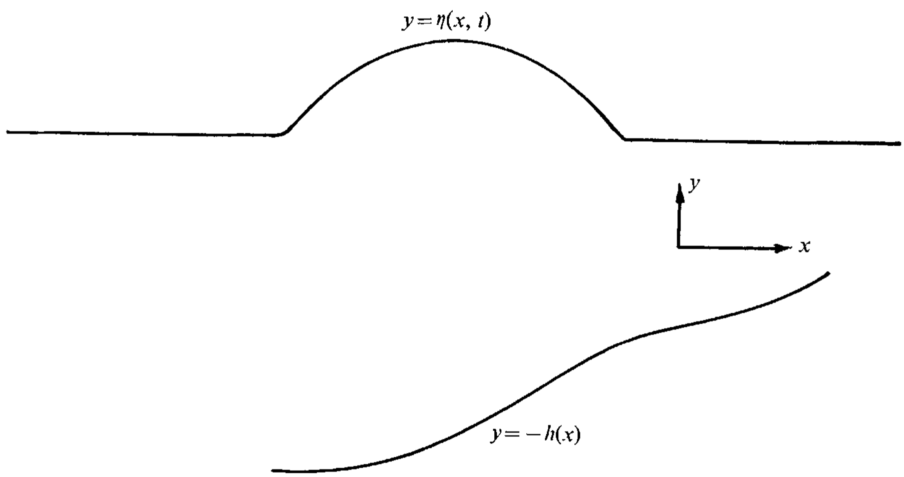

FIGURE 1. Co-ordinate system.

For small ϵ we seek a solution of (2.1) and (2.2) in the form $\phi = \alpha(\phi_0+\epsilon\phi_1+\ldots)$, where α is a measure of the wave amplitude. We find that, to $O(\epsilon)$, ϕ may be expressed in terms of a new unknown function $F(x,t)$ as follows

$$\phi = \alpha(F+\epsilon(-y(hF_x)_x-\tfrac{1}{2}y^2F_{xx})+O(\epsilon^2)). \tag{2.5}$$

Substitution into (2.3) and (2.4) then gives a pair of coupled equations for η and F, both functions of x and t only. However, these will be further simplified as it is well known that the Boussinesq solitary wave may be characterized by requiring α and ϵ to be comparable small quantities (Ursell 1953). Thus we put

$$\eta = \alpha(E+O(\epsilon^2)), \tag{2.6}$$

where $E(x,t)$ is another unknown function, substitute into (2.3) and (2.4) and retain all terms up to $O(\epsilon^3)$ or $O(\alpha\epsilon^2)$, etc. This procedure leads to the Boussinesq equations

$$E+F_t+\tfrac{1}{2}\alpha F_x^2 = 0, \tag{2.7}$$

$$E_t+(hF_x)_x+\alpha(EF_x)_x+\epsilon(\tfrac{1}{3}h^3F_{xx})_{xx}+\epsilon(\tfrac{1}{2}h^2h_{xx}F_x)_x = 0. \tag{2.8}$$

These equations are analogous to those used by Boussinesq (1872) when h is constant, and equivalent versions have been given by Mei & Le Méhauté (1966), and Peregrine (1967). When the terms of $O(\epsilon)$ are omitted they reduce to the non-linear shallow water equations, and when the terms of $O(\alpha)$ are also omitted they reduce to the linearized shallow water equations. Thus they contain the first-order effects of non-linearity, represented by α, and of frequency dispersion represented by ϵ.

It will be useful in the sequel to identify (2.7) and (2.8) as the Euler equations of a certain Lagrangian. Indeed, Whitham (1967) has shown that the Boussinesq equations for constant h may be derived by suitably approximating a certain Lagrangian for the system (2.1) to (2.4), and we shall follow a similar procedure here. Luke (1966*b*) has shown that the system (2.1) and (2.4) can be derived from the variational principle

$$\delta \iint \left\{ \int_{-h}^{\eta} (\phi_t + y + \tfrac{1}{2}\phi_x^2 + \epsilon^{-1}\tfrac{1}{2}\phi_y^2)\, dy \right\} dx\, dt = 0, \tag{2.9}$$

where the infinitesimal variations $\delta\phi$, $\delta\eta$ are sufficiently differentiable and vanish as x, t approach the boundary of the region of integration. If the expansions (2.5) and (2.6) are now substituted into (2.9) the integrand, to $O(\epsilon^3)$ with the omission of certain divergence terms which do not contribute to the Euler equations, is $\alpha^2 L$ where

$$L(E, F_x, F_{xx}, F_t; x) = EF_t + \tfrac{1}{2}E^2 + \tfrac{1}{2}(h + \alpha E)\, F_x^2 - \epsilon\tfrac{1}{6}h^3 F_{xx}^2 + \epsilon\tfrac{1}{4}h^2 h_{xx} F_x^2, \tag{2.10}$$

and L may now be identified as an appropriate Lagrangian for (2.7) and (2.8). Indeed the variation of L with respect to E gives (2.7), and the variation with respect to F gives

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial F_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial F_x}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial F_{xx}}\right) = 0, \tag{2.11}$$

which is just (2.8). The form of (2.11) will be useful in the sequel as it is in conservation form, and corresponds to the explicit absence of F in L. It represents for small ϵ, conservation of mass. Another conservation law may be found from the explicit absence of t in L, and is

$$\frac{\partial}{\partial t}\left(F_t \frac{\partial L}{\partial F_t} - L\right) + \frac{\partial}{\partial x}\left(F_t \frac{\partial L}{\partial F_x} + 2F_{tx}\frac{\partial L}{\partial F_{xx}}\right) - \frac{\partial^2}{\partial x^2}\left(F_t \frac{\partial L}{\partial F_{xx}}\right) = 0. \tag{2.12}$$

This equation represents conservation of energy, and $-\{F_t(\partial L/\partial F_t) - L\}$ may be regarded as an energy density, although it differs from the exact energy density, even for small ϵ, by the previous omission of divergence terms from L. Nevertheless, it may be shown that certain average energy densities can be computed from $F_t(\partial L/\partial F_t) - L$. An equation, which corresponds to (2.11) and (2.12), but represents momentum, is

$$\frac{\partial}{\partial t}\left(F_x \frac{\partial L}{\partial F_t}\right) + \frac{\partial}{\partial x}\left(F_x \frac{\partial L}{\partial F_x} - L + 2F_{xx}\frac{\partial L}{\partial F_{xx}}\right) - \frac{\partial^2}{\partial x^2}\left(F_x \frac{\partial L}{\partial F_{xx}}\right) + \frac{\partial L}{\partial x} = 0, \tag{2.13}$$

where the last term is the explicit derivative of L with respect to x through the dependence of L on h. This is not a true conservation law as it contains the inhomogeneous term $\partial L/\partial x$ which represents the horizontal pressure thrust due to the bottom slope. A further conservation law is

$$\frac{\partial}{\partial t}(F_x)+\frac{\partial}{\partial x}(-F_t) = 0. \tag{2.14}$$

Now that the ordering parameters α, ϵ have served their purpose, we shall, in the following sections, revert to dimensionless co-ordinates based on a length scale h_0 and a velocity scale $(gh_0)^{\frac{1}{2}}$. Thus we shall use (2.7), (2.8) and the subsequent equations, but with $\alpha = \epsilon = 1$, so that e.g. $y = E$ is the equation of the free surface to the approximation considered. We shall also, without any ambiguity, call F the velocity potential and

$$U = F_x \tag{2.15}$$

the velocity.

3. The solitary wave

In this section it will be assumed that h is constant. We shall seek a solution of the Boussinesq equations (2.7) and (2.8) for which E and U are functions only of the phase

$$\theta = \kappa(x-ct), \tag{3.1}$$

where κ (wave-number) and c (wave speed) are constants. Thus we seek a solution of the form

$$E = B+e(\theta), \tag{3.2}$$

$$U = A+u(\theta), \tag{3.3}$$

where A, B are constants, representing the mean velocity and mean height respectively and defined so that e, u and all their derivatives vanish as $|\theta| \to \infty$ (we are anticipating from the form of (2.7) and (2.8) that any such solution will be even in θ). The corresponding form for the potential, F, which must satisfy $U = F_x$ and be consistent with (2.7) is

$$F = Ax-Ct+f(\theta), \tag{3.4}$$

where

$$f(\theta) = \int_0^\theta \kappa^{-1} u(\theta')\, d\theta'$$

and C is a constant, related to the Bernoulli constant.

Substitution of (3.2) and (3.4) into (2.7), and application of the limiting behaviour as $|\theta| \to \infty$ implies that

$$e = c^*u-\tfrac{1}{2}u^2, \tag{3.5}$$

$$C = B+\tfrac{1}{2}A^2, \tag{3.6}$$

where c^* is defined below (3.7). Then substitution of (3.2) and (3.3) into (2.8), elimination of e by (3.5), and two integrations with respect to θ, imply that

$$\tfrac{1}{3}h^3\kappa^2u_\theta^2 = w(u) \equiv (c^{*2}-h^*)\,u^2-c^*u^3+\tfrac{1}{4}u^4, \tag{3.7}$$

where

$$c^* = c-A, \quad h^* = h+B.$$

This has the solution
$$u = \frac{u_m \operatorname{sech}^2 p\theta}{1-d\tanh^2 p\theta}, \tag{3.8}$$
where
$$u_m = 2(c^* - \sqrt{h^*}), \tag{3.9}$$
$$d = \frac{c^* - \sqrt{h^*}}{c^* + \sqrt{h^*}}, \tag{3.10}$$
$$\kappa p = \sqrt{\tfrac{3}{4}}(c^{*2} - h^*)^{\frac{1}{2}} h^{-\frac{3}{2}}, \tag{3.11}$$
and we have selected the origin of θ to be the wave crest, where both e and u achieve their maximum values e_m and u_m respectively. Indeed we find that
$$e_m = \sqrt{(h^*)}\, u_m. \tag{3.12}$$
Thus the solitary wave profile, the wave amplitude e_m and the 'wavelength' $(\kappa p)^{-1}$ are determined completely by the constants A, B and c; the constant κ plays the subsidiary role of relating the x scale to the θ scale. For small values of e_m (h and h^* being $O(1)$), d is $O(e_m)$ and may be neglected, and then (3.8) reduces to the solitary wave profile found by Boussinesq (1871); also the wave speed formula (3.9) is then equivalent to the more commonly quoted formula
$$c^{*2} = h^* + e_m.$$
Although our derivation of the Boussinesq equations was such that, for consistency, all formulae such as (3.8) should be reduced to their lowest order in e_m, we shall continue to work with the 'exact' formulae above ('exact' in the sense that they are exact solutions of the Boussinesq equations (2.7) and (2.8)); indeed it causes no extra algebraic inconvenience to ignore the smallness of e_m, and the retention of the higher order terms in e_m may give some indication of the effects of increasing non-linearity. It may be shown that the profile given by (3.8) is a close approximation to the Boussinesq profile (e.g. the maximum difference is approximately 2 % for $e_m h^{-1} = 0{\cdot}1$, and approximately $7\frac{1}{2}$ % for $e_m h^{-1} = 0{\cdot}4$) and the latter was shown by Daily & Stephan (1952) to be in good agreement with experimentally observed profiles for values of $e_m h^{-1}$ as large as 0·6 (for the larger values of $e_m h^{-1}$ the experimentally observed profile is thinner near the crest than the Boussinesq profile); similarly the observations of Daily & Stephan show that the wave speed given by (3.9) is approximately 6 % too high for $e_m h^{-1} = 0{\cdot}6$, with a decreasing error for decreasing $e_m h^{-1}$.

We shall conclude this section with the calculation of various quantities of interest associated with the solitary wave. First, we give the following definitions. If $P(\theta)$ is the relevant quantity, then its mean is
$$\overline{P} = \lim_{|\theta|\to\infty} \overline{P}(\theta); \tag{3.13}$$
the reduced (or wave) quantity is
$$p(\theta) = P(\theta) - \overline{P}; \tag{3.14}$$
and its wave average (or mean) is
$$\hat{p} = \int_{-\infty}^{\infty} p(\theta)\, d\theta, \tag{3.15}$$

so that $\kappa^{-1}\hat{p}$ is the wave average with respect to the x scale. Clearly $\bar{P}$ is a function of A, B and $\kappa^{-1}\hat{p}$ is a function of A, B and c. Thus we find that

$$\hat{e} = \sqrt{\tfrac{16}{3}}\kappa h^{\frac{3}{2}}(c^{*2}-h^*)^{\frac{1}{2}}, \tag{3.16}$$

$$\hat{u} = \sqrt{\tfrac{16}{3}}\kappa h^{\frac{3}{2}}\cosh^{-1}(c^*/\sqrt{h^*}); \tag{3.17}$$

also the mean Lagrangian, and the wave average Lagrangian are

$$\bar{L} = -BC+\tfrac{1}{2}B^2+\tfrac{1}{2}(h+B)A^2, \tag{3.18}$$

$$\hat{l} = \hat{u}(h^*A-cB)-\hat{w}, \tag{3.19}$$

where $\hat{w}$ is the wave average of the polynomial $w(u)$ defined in (3.7), and is given by

$$\hat{w} = \kappa W, \tag{3.20}$$

where $$W = W(A,B;c,h) = \sqrt{\tfrac{4}{3}}h^{\frac{3}{2}}\int_0^{u_m}\sqrt{(w(u))}\,du. \tag{3.21}$$

It may be noted that $\hat{e}$ is the wave average of the mass density (apart from the constant proportionality factor ρh_0, where ρ is the density of the fluid), so that $\kappa^{-1}\hat{e}$ is the mass carried forward by the wave. Also the wave average of the momentum density (apart from the factor $\rho h_0\sqrt{(gh_0)}$) is

$$h^*\hat{u}+A\hat{e}+\partial\hat{w}/\partial c. \tag{3.22}$$

Finally, the wave average of the energy density (apart from the factor $\rho g h_0^2$) is

$$C\hat{e}+Ah^*\hat{u}+\left(c\frac{\partial\hat{w}}{\partial c}-\hat{w}\right), \tag{3.23}$$

and further $$c(\partial\hat{w}/\partial c)-\hat{w} = (\tfrac{4}{3})^{\frac{3}{2}}\kappa h^{\frac{3}{2}}(c^{*2}-h^*)^{\frac{3}{2}}+A(\partial\hat{w}/\partial c). \tag{3.24}$$

4. Modulations caused by slowly varying depth

It will now be supposed that h is a function of x but is *slowly varying* in the sense that h varies little over a distance comparable with the length of the wave. Thus we shall assume that $h = h(X)$ where

$$X = \beta x, \quad T = \beta t, \tag{4.1}$$

and β is a small parameter such that $\beta \ll \kappa p$. In this section we shall find equations which govern the modulations to the solitary wave of §3 caused by this slow variation of the depth. This will be achieved by finding an asymptotic solution of the Boussinesq equations which represents a slowly varying solitary wave i.e. locally this asymptotic solution may be represented by the uniform solution of §3, but the parameters A, B, C, c and κ which determine that solution are now slowly varying and so functions of X, T. Our principal aim is the determination of transport equations for these parameters. Whitham (1965*a*, *b*) has considered problems of this type for periodic slowly varying wave trains governed by non-linear, dispersive equations. The procedures described in this section are closely related to the procedures developed by Whitham and other workers in this field.

Thus we are motivated to seek an asymptotic solution of the Boussinesq equations (2.7) and (2.8) of the form

$$\left.\begin{aligned} E &= B(X,T)+e(\theta;X,T)+\beta E_1(\theta;X,T)+O(\beta^2),\\ U &= A(X,T)+u(\theta;X,T)+\beta U_1(\theta;X,T)+O(\beta^2).\end{aligned}\right\} \tag{4.2}$$

A, B are determined so that e, u and all their derivatives with respect to θ vanish as $|\theta| \to \infty$, and the phase θ is such that

$$\theta_x = \kappa, \quad \theta_t = -\kappa c, \tag{4.3}$$

and so $$\theta = \beta^{-1}\Theta(X,T), \quad \text{where} \quad \kappa = \Theta_X, \; -\kappa c = \Theta_T. \tag{4.4}$$

θ is a *fast* variable, which has yet to be determined, and X, T are *slow* variables; (4.2) is a two-scale asymptotic expansion of a type familiar in the context of ordinary differential equations. Since derivatives with respect to θ are $O(1)$, while derivatives with respect to X and T are $O(\beta)$, it is clear that when (4.2) is substituted into (2.7) and (2.8), the terms of $O(1)$ are just those which describe the solitary wave of §3 and so e, u are determined as functions of θ by (3.5) to (3.11), except that the parameters A, B, C, c and κ are now functions of X, T. The transport equations which determine these parameters are found by applying the principle that the asymptotic expansion (4.2) is to be uniformly valid i.e. βE_1 and βU_1 are $O(\beta)$ with respect to $B+e$ and $A+u$ respectively for all θ. Thus we shall assume that E_1 and U_1 can be constructed so that

$$A_1^{\pm} = \lim_{\theta\to\pm\infty} U_1, \quad B_1^{\pm} = \lim_{\theta\to\pm\infty} E_1 \tag{4.5}$$

exist, and all derivatives of U_1 and E_1 with respect to θ vanish as $|\theta| \to \infty$. It will be shown in subsection (*a*) below that such a construction is indeed possible. From (4.5) we define

$$A_1 = \tfrac{1}{2}(A_1^+ + A_1^-), \quad [U_1] = (A_1^+ - A_1^-), \quad u_1 = U_1 - A_1, \tag{4.6}$$

with similar definitions for B_1, $[E_1]$ and e_1.

Next we seek an asymptotic expansion for the potential F such that $U = F_x$ where U is given by (4.2). Thus

$$F = \beta^{-1}\psi(X,T) + f(\theta;X,T) + \psi_1(X,T) + \beta f_1(\theta;X,T) + \ldots, \tag{4.7}$$

where the remaining terms are $O(\beta^2)$ if they involve θ and $O(\beta)$ otherwise, and

$$\psi_X = A, \quad \psi_T = -C, \tag{4.8}$$

$$f = \int_0^\theta \kappa^{-1} u(\theta';X,T)\,d\theta', \tag{4.9}$$

$$\psi_{1X} = A_1, \quad \psi_{1T} = -C_1, \tag{4.10}$$

$$\kappa f_{1\theta} = u_1 - f_X. \tag{4.11}$$

It follows that

$$F_t = -C - cu + \beta(-C_1 - cu_1 + f_T + cf_X) + O(\beta^2). \tag{4.12}$$

Then substitution of (4.2) and (4.12) into (2.7) gives, for the terms of $O(1)$, (3.5) and (3.6), while the term of $O(\beta)$ is

$$B_1 + e_1 - C_1 - cu_1 + f_T + cf_X + (A+u)(A_1+u_1) = 0. \tag{4.13}$$

Letting $\theta \to \pm\infty$ we find that

$$C_1 = B_1 + AA_1, \tag{4.14}$$

$$[E_1] = c^*[U_1] - (\kappa^{-1}\hat{u})_T - c(\kappa^{-1}\hat{u})_X. \tag{4.15}$$

Next, substitution of (4.2) and (4.12) into the consistency relation (2.14) yields, for the term of $O(\beta)$,

$$A_T + C_X + \{\kappa_T + (\kappa c)_X\} u = 0; \tag{4.16}$$

and letting $|\theta| \to \infty$ we have

$$A_T + C_X = 0, \tag{4.17}$$

whence

$$\kappa_T + (\kappa c)_X = 0. \tag{4.18}$$

These two equations are just the consistency relation for ψ and θ respectively and provide two transport equations. A third is (3.6); two more are needed and may now be determined in each of three ways.

(*a*) *Direct method*

In this subsection the transport equations will be found by first finding U_1 (and hence E_1) explicitly. The methods used here are similar to those used for slowly varying periodic wave trains by Luke (1966*a*) for a Klein–Gordon equation, and Hoogstraten (1968) for the Korteweg–de Vries equation, and for the Boussinesq equations of constant depth, and are analogous to the Poincaré technique for ordinary differential equations (e.g. in particular to the work of Kuzmak 1959).

If (4.2) is substituted into (2.8), then the term of $O(1)$ defines the solitary wave of §3, while the term of $O(\beta)$ gives

$$\kappa\{-cE_1 + (h^* + e)U_1 + (h + E_1)(A + u) + \tfrac{1}{3}\kappa^2 h^3 U_{1\theta\theta} + h^3\kappa_X u_\theta + h^3\kappa u_{\theta X} + 2h^2 h_X \kappa u_\theta\}_\theta$$
$$+ \{B + e\}_T + \{(h^* + e)(A + u)\}_X = 0. \tag{4.19}$$

Letting $|\theta| \to \infty$ we see that

$$B_T + (h^*A)_X = 0, \tag{4.20}$$

which is the fourth transport equation. Then (4.19) is integrated with respect to θ, and after elimination of e and e_1 by (3.5) and (4.13) respectively, we find that

$$\tfrac{1}{3}h^3\kappa^2 u_{1\theta\theta} - (c^{*2} - h^*)u_1 + 3c^* u u_1 - \tfrac{3}{2}u^2 u_1 = G, \tag{4.21}$$

where

$$G = D_1 - \kappa^{-1}\int_0^\theta \{e_T + (h^*u)_X + (Ae)_X + (eu)_X\}\,d\theta'$$
$$- h^3\kappa_X u_\theta - h^3\kappa u_{\theta X} - 2h^2 h_X \kappa u_\theta$$
$$- (c^* - u)(f_T + cf_X) - uB_1 - 2uc^*A_1 + \tfrac{3}{2}u^2 A_1, \tag{4.22}$$

where $D_1(X, T)$ is a 'constant' of integration. It may now be observed, by differentiating (3.7) twice with respect to θ, that the homogenous part of (4.21) (i.e. when G is replaced by zero) has the solution $u_1 = u_\theta$. Thus (4.21) may be integrated again with respect to θ after first multiplying by u_θ, and we find that

$$\tfrac{1}{3}h^3\kappa^2(u_{1\theta}u_\theta - u_1 u_{\theta\theta}) = \int_{-\infty}^\theta u_\theta G\,d\theta' + D_2, \tag{4.23}$$

where $D_2(X, T)$ is another 'constant' of integration. Letting $\theta \to \pm\infty$ we see that the left-hand side then vanishes, and so therefore must the right-hand side. Thus $D_2 = 0$ and

$$\int_{-\infty}^{\infty} u_\theta G\, d\theta = 0. \tag{4.24}$$

Since u is an even function of θ, (4.24) involves only A, B, c and κ and is the fifth transport equation. The complete set of transport equations is thus (3.6), (4.17), (4.18), (4.20) and (4.24). One further integration of (4.23) yields

$$\tfrac{1}{3}h^3\kappa^2 u_1 = \left(D_3 + \int_0^\theta H(u_\theta)^{-2}\, d\theta'\right) u_\theta, \tag{4.25}$$

where
$$H = \int_{-\infty}^{\theta} u_\theta G\, d\theta',$$

and $D_3(X, T)$ is another 'constant' of integration. It may now be shown that Hu^{-1} remains finite as $\theta \to \pm\infty$ (in spite of the fact that e.g. u_X contains terms of the type θu_θ), and so u_1 remains bounded as $\theta \to \pm\infty$, and all its derivatives vanish as $\theta \to \pm\infty$. Of course, u_1 is determined by (4.25) as a function of θ only, and still depends on the 'unknown' constants A, B_1, D_1 and D_3; these may presumably be determined in a similar way to the above by continuing the asymptotic expansion (4.2) to a higher order in β.

(b) *Averaged conservation laws*

In this subsection the transport equations will be derived by applying suitable averaging procedures to the conservation laws (2.11), (2.12), (2.13) and (2.14). These procedures are analogous to those used by Whitham (1965*a*) for slowly varying periodic wave trains, and are related to the Krylov–Boguliobov technique familiar in the context of ordinary differential equations.

The typical conservation law has the form

$$\partial P/\partial t + \partial Q/\partial x + \beta R = 0, \tag{4.26}$$

where R is proportional to h_X and its presence is due to the inhomogeneity of the medium. Since E, U have asymptotic expansions of the form (4.2), it follows that P, Q, R have similar expansions e.g.

$$P = P_0(\theta; X, T) + \beta P_1(\theta; X, T) + O(\beta^2). \tag{4.27}$$

Then our hypotheses on E, U are such that

$$P_i^{\pm} = \lim_{\theta\to\pm\infty} P_i \quad (i = 0, 1) \tag{4.28}$$

certainly exist, and we define

$$\bar{P}_i = \tfrac{1}{2}(P_i^+ + P_i^-), \quad [P_i] = (P_i^+ - P_i^-) \quad (i = 0, 1). \tag{4.29}$$

Since P_0, etc., are even in θ, $[P_0]$, etc., vanish but as we shall see, $[P_1]$, etc., in general are not zero. Also we observe that

$$\bar{P}_i = \lim_{\gamma\to\infty} \frac{1}{2\gamma}\int_{-\gamma}^{\gamma} P_i\, d\theta. \tag{4.30}$$

Next we define the reduced (or wave) quantity by

$$p = P_0 - \bar{P}_0 \tag{4.31}$$

and its wave average (or mean) by

$$\hat{p} = \lim_{\gamma\to\infty} \int_{-\gamma}^{\gamma} p\, d\theta. \tag{4.32}$$

We now substitute the expansions such as (4.27) into (4.26) and equate to zero the term of $O(1)$, and the term with coefficient β; the former gives a relation satisfied identically by the solitary wave of § 3, and the latter gives

$$\bar{P}_{0T} + \bar{Q}_{0X} + \bar{R}_0 + p_T + q_X + r - \kappa c P_{1\theta} + \kappa Q_{1\theta} = 0. \tag{4.33}$$

First we take the mean of (4.33), i.e. the averaging procedure defined by (4.30). This yields the equation

$$\bar{P}_{0T} + \bar{Q}_{0X} + \bar{R}_0 = 0. \tag{4.34}$$

Next we substract (4.34) from (4.33), and take the wave average, i.e. the averaging procedure defined by (4.32). This yields the equation

$$(\hat{p})_T + (\hat{q})_X + \hat{r} - \kappa c[P_1] + \kappa[Q_1] = 0. \tag{4.35}$$

Equations (4.34) and (4.35) are transport equations for A, B, C, c and κ, and also for $[E_1]$ and $[U_1]$.

It is convenient when applying the averaged conservation laws (4.34) and (4.35) to do so in conjunction with the formulae (3.5), (3.6) and (4.15), all of which are derived from (2.7) (which is not in conservation form). If (4.34) is applied to (2.14) and (2.11), then we obtain (4.17) and (4.20) respectively; further applications to (2.12) and (2.13) yield two transport equations for A, B, C which are equivalent to (4.17) and (4.20). Application of (4.35) to (2.14) yields the transport equation (4.18); application of (4.35) to (2.11) yields

$$(\hat{e})_T + \left(h^*\hat{u} + A\hat{e} + \frac{\partial \hat{w}}{\partial c}\right)_X + \kappa(-c^*[E_1] + h^*[U_1]) = 0, \tag{4.36}$$

where $\hat{w}$ is defined by (3.20), (3.21). $[E_1]$ and $[U_1]$ may now be found in terms of A, B, C, c and κ by solving (4.15) and (4.36) simultaneously. Next application of (4.35) to (2.12) and subsequent elimination of $[E_1]$ and $[U_1]$ yields

$$\left(c\frac{\partial \hat{w}}{\partial c} - \hat{w}\right)_T + \left(c\left(c\frac{\partial \hat{w}}{\partial c} + \hat{w}\right)\right)_X - \frac{\partial \hat{w}}{\partial T} = 0; \tag{4.37}$$

similarly application of (4.35) to (2.13) and subsequent elimination of $[E_1]$ and $[U_1]$ yields

$$\left(\frac{\partial \hat{w}}{\partial c}\right)_T + \left(c\frac{\partial \hat{w}}{\partial c}\right)_X - \frac{\partial \hat{w}}{\partial X} = 0, \tag{4.38}$$

which is easily seen to be equivalent to (4.37). In both of these equations $\hat{w} = \kappa W(c; X, T)$ so that e.g. $\partial\hat{w}/\partial T$ means differentiation with respect to T while c and κ are kept constant; $\hat{w}$ depends on X, T through its dependence on A, B and h. Finally, it may be shown that (4.24) can be reduced to either of (4.37) or (4.38).

(c) *Averaged variational principle*

Whitham (1965*a*, 1967) (see also Bretherton 1968) has developed an heuristic procedure for finding the transport equations for slowly varying periodic wave trains, when the governing equations are the variational equations of a Lagrangian density. Briefly this procedure consists of calculating the average value over one period of this Lagrangian density for the uniform wave train, which itself depends on a set of parameters such as frequency, wave-number, etc.; this averaged Lagrangian is then subjected to the variation of these parameters.

The Boussinesq equations (2.7) and (2.8) for constant h possess a solution of the form (3.2), (3.3) and (3.4) which has a period γ, where now u has zero mean so that

$$A = \frac{1}{2\gamma}\int_{-\gamma}^{\gamma} U\,d\theta, \tag{4.39}$$

with a similar equation for B, and

$$\tfrac{1}{3}h^3\kappa^2u_\theta^2 = K_1+K_2u+(B+\tfrac{1}{2}A^2-C)\,u^2+w(u) \equiv v(u); \tag{4.40}$$

$w(u)$ is defined by (3.7), and K_1, K_2 are constants of integration. If the polynomial $v(u)$ has the four real zeros $d^{-1}u_m > u_m > u_1 > u_2$, we select that solution of (4.40) for which u lies between u_1 and u_m. Then if K_1, $K_2 \to 0$ simultaneously, so that the period $\gamma \to \infty$, the solution of (4.40) becomes the solitary wave (3.8). The averaged Lagrangian is defined to be

$$\mathscr{L} = \frac{1}{2\gamma}\int_{-\gamma}^{\gamma} L\,d\theta \tag{4.41}$$

and is given by

$$\mathscr{L} = \bar{L}(A,C;B)+\tfrac{1}{2}K_1-\gamma^{-1}\surd\tfrac{4}{3}\kappa h^{\frac{3}{2}}\int_{u_1}^{u_m}\surd(v(u))\,du, \tag{4.42}$$

where $\bar{L}$ is defined by (3.18). $\mathscr{L}$ is thus a function of the parameters A, B, C; K_1, K_2, ω ($=\kappa c$ the frequency) and κ. For a slowly varying wave train these parameters are functions of X, T, and Whitham's procedure is to subject $\mathscr{L}$ to variations of ψ (where $\psi_X = A$, $\psi_T = -C$), Θ (where $\Theta_X = \kappa$, $\Theta_T = -\omega$), B, K_1 and K_2. Thus the transport equations are

$$\left(\frac{\partial\mathscr{L}}{\partial C}\right)_T - \left(\frac{\partial\mathscr{L}}{\partial A}\right)_X = 0, \tag{4.43}$$

$$\left(\frac{\partial\mathscr{L}}{\partial \omega}\right)_T - \left(\frac{\partial\mathscr{L}}{\partial \kappa}\right)_X = 0, \tag{4.44}$$

$$\frac{\partial\mathscr{L}}{\partial B} = 0, \tag{4.45}$$

$$\frac{\partial\mathscr{L}}{\partial K_1} = 0, \tag{4.46}$$

$$\frac{\partial\mathscr{L}}{\partial K_2} = 0, \tag{4.47}$$

Equation (4.46) is the dispersion relation which determines γ as a function of the parameters, and (4.47) is the condition that u have zero mean. Two more transport equations are obtained by applying the consistency relations (4.17) and (4.18). Altogether there are seven transport equations for the seven parameters. Now we can let $K_1, K_2 \to 0$, so that $\gamma \to \infty$ and $\mathscr{L} \to \bar{L}$; (4.43) and (4.45) become

$$\left.\begin{aligned} \left(\frac{\partial \bar{L}}{\partial C}\right)_T - \left(\frac{\partial \bar{L}}{\partial A}\right)_X &= 0, \\ \frac{\partial \bar{L}}{\partial B} &= 0, \end{aligned}\right\} \tag{4.48}$$

which are just (4.20) and (3.6) respectively. Equation (4.44) becomes

$$\left(\frac{\partial \hat{w}}{\partial \omega}\right)_T - \left(\frac{\partial \hat{w}}{\partial \kappa}\right)_X = 0, \tag{4.49}$$

where $\hat{w}$ is regarded as a function of ω, κ and X, T; if instead $\hat{w}$ is regarded as a function of c, κ and X, T then (4.49) is just (4.37), or (4.38). Equations (4.46) and (4.47) do not retain any significance as $\gamma \to \infty$. The form of (4.48) and (4.49) shows that our transport equations can be derived from two variational principles; first by subjecting $\bar{L}$, a function of A, B, C, to variations of ψ and B; and secondly by subjecting $\hat{w}$, a function of $\omega(=\kappa c)$, κ and X, T (through A, B and h), to variations of Θ.

5. Solution of the transport equations

The transport equations are (3.6), (4.17), (4.20), (4.18) and (4.37), and are displayed here again for convenience:

$$C = B + \tfrac{1}{2}A^2, \tag{5.1}$$

$$A_T + C_X = 0, \tag{5.2}$$

$$B_T + (A(h+B))_X = 0, \tag{5.3}$$

$$K_T + (\kappa c)_X = 0, \tag{5.4}$$

$$\left(c\frac{\partial \hat{w}}{\partial c} - \hat{w}\right)_T + \left(c\left(c\frac{\partial \hat{w}}{\partial c} - \hat{w}\right)\right)_X + \frac{\partial \hat{w}}{\partial T} = 0. \tag{5.5}$$

The first three equations involve only A, B and C; they are, perhaps not unexpectedly, just the shallow-water equations (i.e. (2.7) and (2.8) with the dispersive terms absent), and can, in principle, be solved. In particular if A and B vanish at $T = 0$ for all X, then they vanish for all T. In any event, A and B can be regarded as known when considering (5.4) and (5.5). Since $\hat{w} = \kappa W$, and W is a function only of c and X, T (through A, B and h), it is convenient to eliminate κ from (5.5):

$$\left(c\frac{\partial W}{\partial c} - W\right)_T + c\left(c\frac{\partial W}{\partial c} - W\right)_X + \frac{\partial W}{\partial T} = 0. \tag{5.6}$$

This is a single equation for c, or better, for

$$V = c\frac{\partial W}{\partial c} - W \tag{5.7}$$

and its general solution can, in principle, be obtained.

We shall now consider a special case when (5.6) may be integrated explicitly. It will be supposed that $h = 1$ for all $X \leqslant 0$, and so the wave evolves from a region where it is uniform. Thus the transport equations are to be solved subject to the initial values, $A = B = 0$ and κ, c constant. (This cannot be exactly true as the solitary wave is infinite in extent, and even when its crest is over a large negative value of X, part of the wave is interacting with the varying h in $X > 0$; however, it is reasonable to suppose that this interaction can be made as small as we please by taking the initial values to be those at an indefinitely large negative value of X.) Thus $A \equiv B \equiv 0$, and since W then depends only on c and h, $\partial W/\partial T \equiv 0$ and (5.6) becomes

$$V_T + cV_X = 0 \tag{5.8}$$

where from (5.7) $c = c(V, X)$. The general solution of (5.8) is

$$V = M(T_0),$$

where
$$T_0 = T - \int_0^X ds\{c(M(T_0), s)\}^{-1}. \tag{5.9}$$

V is therefore an 'adiabatic invariant', i.e. it is constant on the wavelet which passed $X = 0$ at a time T_0 and is travelling with speed $c(M(T_0), X)$. In general (5.9) contains the possibility of shock formation at those places where T_0 cannot be found as a function of X, T. However, since κ, c are initially constant, so is V and the solution of (5.8) required is just V equals a constant (i.e. M is a constant). We note that since A and B are zero, it follows from (3.23) that V is the wave-average of the energy density with respect to the x scale, and so the solution we have obtained is just that which preserves the energy of the wave. This of course, might have been expected, as our asymptotic expansion is one which ignores reflexions and there is no other outlet for the loss of energy. Further, it follows from (3.24) that

$$c^2 = h + Nh^{-1}, \tag{5.10}$$

where N is a constant (in general N is a function of T_0). The wave amplitude is found from (3.9) and (3.12), and is

$$e_m = 2((h^2+N)^{\frac{1}{2}} - h). \tag{5.11}$$

Figure 2 shows a plot of $e_m/(e_m)_0$ against h where $(e_m)_0$ is the value of e_m at $h = 1$ (i.e. $X = 0$); it exhibits the fact that $e_m/(e_m)_0$ for each N, increases as h decreases, but, for each h, decreases as N (and hence $(e_m)_0$) increases. Also shown is the graph of $h^{-0\cdot47}$ which represents the results of Ippen & Kulin's (1955) experiments on the behaviour of a solitary wave on a beach of constant slope 0·023; they observed a fairly wide scatter, and the curve shown is a best fit for several observations with values of $(e_m)_0$ ranging from 0·2 to 0·7 (and also with varying values for the initial depth of fluid). They also observed a small decrease in amplitude at the foot of the beach, where, in the experimental set up, there was an abrupt change in beach slope from zero to 0·023; this was presumably due to a reflexion. We have ignored this initial energy loss in displaying their results on figure 2 by allowing the graph of the experimental points (viz. $h^{-0\cdot47}$) to pass through $h = 1$ when $e_m = (e_m)_0$ whereas the true curve would be similar in shape but displaced downwards by a small amount. For small values of $(e_m)_0$ we have

$$e_m/(e_m)_0 \approx h^{-1}, \tag{5.12}$$

an approximation which is accurate to within 5 % for $(e_m)_0 = 0{\cdot}01$, $0{\cdot}3 \leqslant h \leqslant 1$ and also for $(e_m)_0 = 0{\cdot}1$, $0{\cdot}6 \leqslant h \leqslant 1$ but becomes increasingly inaccurate for larger values of $(e_m)_0$.

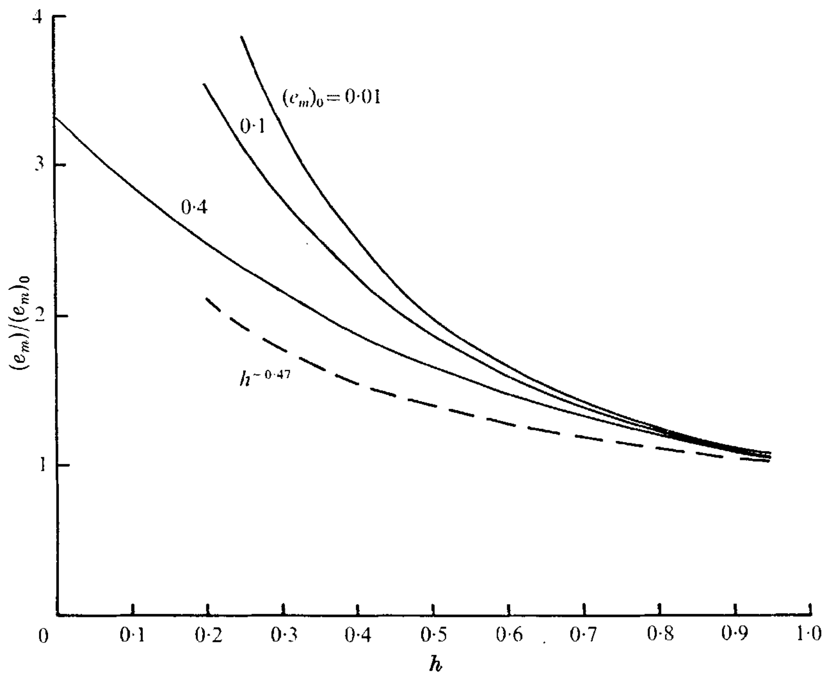

FIGURE 2. Graph of $(e_m)/(e_m)_0$ against h.

Other properties of the wave can also be determined from (5.10). Thus we find that

$$\kappa p = \sqrt{\left(\frac{3N}{4}\right)} h^{-2}, \tag{5.13}$$

$$\kappa^{-1}\hat{e} = \sqrt{\left(\frac{16N}{3}\right)} h, \tag{5.14}$$

$$\kappa^{-1}\hat{u} = \sqrt{(\tfrac{16}{3})}\, h^{\frac{3}{2}} \cosh^{-1}(1+Nh^{-2})^{\frac{1}{2}}, \tag{5.15}$$

$$N[E_1] = \tfrac{3}{2}h^{\frac{1}{2}}h_X(h^2+N)^{\frac{1}{2}}(\kappa^{-1}\hat{u}) + \sqrt{\left(\frac{16N}{3}\right)} N h_X, \tag{5.16}$$

$$N[U_1] = \tfrac{3}{2}h_X(h+Nh^{-1})(\kappa^{-1}\hat{u}). \tag{5.17}$$

Equation (5.13) shows that the length of the wave, $(\kappa p)^{-1}$ decreases as h decreases; (5.14) to (5.17) show that the mass contained in the wave is not conserved, and is fed into a mean flow, and a change in the mean depth, both proportional to βh_X. Further the equations (5.16) and (5.17) indicate that the effect of the terms of $O(\beta)$ in the asymptotic expansion (4.2) is to cause increasing asymmetry due to steepening on the front face, and flattening on the rear face. If we adopt the criterion that the wave will break when $u_m = c$ (i.e. the velocity at the crest equals the wave velocity) then (5.10) implies that the wave will break when $h = \sqrt{(\frac{1}{3}N)}$; at this value of h, $e_m h^{-1} = 2$ which is much greater than the most commonly accepted theoretical value of 0·78 for the highest wave on a constant depth (McCowan 1894), although Ippen & Kulin's experiments showed that $e_m h^{-1} \approx 1{\cdot}2$ at the breaking depth for a beach slope of 0·023. Of course the value $u_m = c$ is almost certainly outside the range of validity of the Boussinesq equations. Finally, from (5.4) we see that κc is constant, and this determines κ.

6. Error estimate

The procedures outlined in §4 have enabled us to construct functions

$$\tilde{E} = B + e + \beta E_1, \quad \tilde{F} = \beta^{-1}\psi + f + \psi_1 + \beta f_1, \quad \tilde{U} = \tilde{F}_x, \tag{6.1}$$

which satisfy the Boussinesq equations (2.7) and (2.8) approximately, with an error of $O(\beta^2)$. That is if

$$D_1(E, F) \equiv \partial L / \partial E, \tag{6.2}$$

$$D_2(E, F) \equiv \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial F_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial F_x}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial F_{xx}}\right), \tag{6.3}$$

where $L(E, F_x, F_{xx}, F_t; X)$ is defined by (2.10) (with $\alpha = \epsilon = 1$) then

$$D_1(\tilde{E}, \tilde{F}) = O(\beta^2), \quad D_2(\tilde{E}, \tilde{F}) = O(\beta^2). \tag{6.4}$$

We now pose the problem: does there exist an exact solution E, F, $U = F_x$ of the Boussinesq equations for which $\tilde{E} - E$ and $\tilde{U} - U$ are $O(\beta^2)$? The following analysis provides a partial answer to this problem.

For simplicity, it will be supposed that $h = 1$ for $X \leqslant 0$ and h takes another constant value for X large and positive and that h is as smooth as desired. Then we may assume that B and ψ vanish for all x and t, and that, from (5.16) and (5.17), $[E_1]$ and $[U_1]$ vanish for sufficiently large $|X|$. Also we can assume that B_1 and ψ_1 vanish for all x and t as their values were not relevant in the construction of E_1 and f_1. Thus the functions defined by (6.1) have been constructed so that $\tilde{E}$, $\tilde{U}$ with all their derivatives, vanish as $|x| \to \infty$, for some time interval $0 \leqslant t \leqslant t_0$. Let E, F be that exact solution of the Boussinesq equations which agrees with $\tilde{E}$, $\tilde{F}$ at $t = 0$, so that

$$E - \tilde{E}|_{t=0} = 0, \quad F - \tilde{F}|_{t=0} = 0. \tag{6.5}$$

We shall now assume that for the initial values (6.5) there exists an exact solution of the Boussinesq equations over the time interval $0 \leqslant t \leqslant t_0$, such that E, U with all their derivatives, vanish as $|x| \to \infty$. Given this, we shall now show that $\tilde{E}$, $\tilde{U}$ differ from E, U by terms of $O(\beta^2)$. Let

$$E' = E - \tilde{E}, \quad F' = F - \tilde{F}, \quad U' = F_x - \tilde{F}_x; \tag{6.6}$$

then

$$D_1(E', F') = -U'\tilde{U} + O(\beta^2), \tag{6.7}$$

$$D_2(E', F') = -(E'\tilde{U} + U'\tilde{E})_x + O(\beta^2), \tag{6.8}$$

where the terms $O(\beta^2)$ are uniform for all x, and $0 \leqslant t \leqslant t_0$. Now if

$$D_3(E,F) \equiv \left(F_t \frac{\partial L}{\partial F_t} - L\right)_t + \left(F_t \frac{\partial L}{\partial F_x} + 2F_{tx}\frac{\partial L}{\partial F_{xx}}\right)_x - \left(F_t \frac{\partial L}{\partial F_{xx}}\right)_{xx} \quad (6.9)$$

then
$$D_3(E,F) \equiv -E_t D_1(E,F) + F_t D_2(E,F). \quad (6.10)$$

It was remarked in §2 that $D_3 = 0$ is the equation for conservation of energy, and that

$$\mathscr{E}(E,F) \equiv L - F_t \partial L/\partial F_t \equiv \tfrac{1}{2}E^2 + \tfrac{1}{2}(h+E)\,U^2 - \tfrac{1}{6}h^3U_x^2 + \tfrac{1}{4}h^2h_{xx}\,U^2 \quad (6.11)$$

may be regarded as an energy density; although it is not positive definite, it may be assumed that it takes only positive values in the long wave approximation being used here (e.g. $|hU_x| \ll |U|$), and that its vanishing implies that E and U vanish. Then, using (6.7), (6.8) and (6.10), it follows that

$$D_3(E',F') = I + O(\beta^2), \quad (6.12)$$

where
$$I = (E' + \tfrac{1}{2}U'^2)\,(E'\tilde{U} + U'\tilde{E})_x - U'\tilde{U}(hU' + E'U' + (\tfrac{1}{3}h^3U'_x)_x)_x. \quad (6.13)$$

On integrating (6.12) with respect to x, we find that

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \mathscr{E}(E',F')\,dx = \int_{-\infty}^{\infty} I\,dx + O(\beta^2). \quad (6.14)$$

Clearly, using integration by parts where necessary and the long wave approximation, the integral of I can be estimated in terms of the integral of $\mathscr{E}$, so that

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \mathscr{E}(E',F')\,dx \leqslant K\int_{-\infty}^{\infty} \mathscr{E}(E',F')\,dx + Q\beta^2, \quad (6.15)$$

where K, Q are constants. Since $\mathscr{E}(E',F')$ vanishes when $t = 0$ it follows that

$$\int_{-\infty}^{\infty} \mathscr{E}(E',F')\,dx \leqslant \beta^2 QK^{-1}(e^{Kt_0} - 1), \quad (6.16)$$

from which we may deduce that

$$E - \tilde{E} = O(\beta^2), \quad U - \tilde{U} = O(\beta^2). \quad (6.17)$$

Since $\tilde{E}$, $\tilde{F}$ contain no reflected terms, (6.17) shows that any reflected energy is $O(\beta^2)$. Indeed this same argument could be used to show that if $\tilde{E}$, $\tilde{F}$ were such that the error in (6.4) was $O(\beta^N)$ for arbitrarily large N, then the reflected energy is also $O(\beta^N)$.

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